Notes on arithmetic of complex random variables

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1 Introduction

1.1 Context

This document summarizes the research done on the arithmetic of complex random variables during my internship in the DSB group, Department of Informatics (IFI). This establishes some theoretical background needed for the implementation of the statistical analysis of complex interval arithmetic in the *Complex Interval Arithmetic Toolbox*, in MATLAB (`https://github.com/unioslo-mn/ifi-complex-interval-arithmetic`). For more context, see `https://perso.eleves.ens-rennes.fr/people/gauvain.thomas/`.

The content of is document is almost fully independent by itself, but relates to complex interval arithmetic for its motivation.

1.2 Related works

- N. O'Donoughue and J. M. F. Moura, "On the Product of Independent Complex Gaussians," in IEEE Transactions on Signal Processing, vol. 60, no. 3, pp. 1050-1063, March 2012, doi: 10.1109/TSP.2011.2177264.
- `https://handwiki.org/wiki/Product_distribution`
- Kurz, Gerhard & Gilitschenski, Igor & Hanebeck, Uwe. (2014). Efficient Evaluation of the Probability Density Function of a Wrapped Normal Distribution. 10.13140/2.1.3082.2084.
- Berens, P. (2009). CircStat: A MATLAB Toolbox for Circular Statistics. Journal of Statistical Software, 31(10), 1–21. https://doi.org/10.18637/jss.v031.i10

2 Change of coordinates

Let R be a random variable with a probability density function $f_R(r)$ and a cumulative distribution function $F_R(r)$.

Let Θ be a random variable with a probability density function $f_\Theta(\theta)$ and a cumulative distribution function $F_\Theta(\theta)$.

$Z = Re^{j\Theta}$ is a random variable with a probability density function $f_Z(z)$ and a cumulative distribution function $F_Z(z)$.

We can compute the cumulative distribution function of Z (as the cumulative distribution function of complex random variables is defined as via the joint distribution of the real and imaginary parts) :

$$F_Z(z) = \mathbf{P}(Re(Z) \leq Re(z), Im(Z) \leq Im(z))$$

And we have :

$$f_{Z;x,y}(x+iy) = \frac{\partial^2 F_{Z;x,y}(x+iy)}{\partial x \partial y}$$

$$f_{Z;x,y}(x+iy) = \frac{1}{r} f_{Z;r,\theta}(\sqrt{x^2+y^2}, \arctan(y/x))$$

3 Defining probability distributions on intervals

3.1 Uniform distributions

Complex uniform distributions are already well defined on any kind of set, thus on any complex interval.

Let X be a uniform complex random variable on S_X with a probability density function $f_X(x) = \frac{1}{|S_X|}$ and Y be a uniform complex random variable on S_Y with a probability density function $f_Y(y) = \frac{1}{|S_Y|}$, with $|S_X|$ (resp. $|S_Y|$) the area of S_X (resp. S_Y).

Then their sum $Z = X + Y$ is defined on the Minkowski sum of the two shapes, which can computed using Interval Arithmetic (IA) with adapted shapes.

Thus we can compute the pdf of Z as follows :

As we know, the pdf of Z is the convolution of the pdfs of X and Y :

$$f_Z(z) = \int_{S_X} f_X(x) f_Y(z-x)dx = \int_{S_X} \frac{\mathbf{1}_{S_X}(x)}{|S_X|}\frac{\mathbf{1}_{S_Y}(z-x)}{|S_Y|}dx = \frac{1}{|S_X||S_Y|}\int_{S_X} \mathbf{1}_{S_Y}(z-x)dx$$

and $\int_{S_X} \mathbf{1}_{S_Y}(z-x)dx$ exactly corresponds to the area of the intersection of S_X and S_Y translated by z, which is the backtracked region of Z.

3.2 Normal distributions

Defining complex normal distributions is just a special case of a multivariate normal distribution.

`https://en.wikipedia.org/wiki/Complex_normal_distribution`

`https://en.wikipedia.org/wiki/Multivariate_normal_distribution`

However, as we consider to define it on a closed bounded area, we need to consider a truncated version of the distribution, so that its support corresponds with the interval.

`https://en.wikipedia.org/wiki/Truncated_normal_distribution`

Formally, it's only a normalization of the probability distribution on the new support.

However, for practical purpose we often use a normal distribution centered in the interval, with bounds further than three standard deviations, which is very closely approximated by a usual normal distribution.

3.3 Normal-like distributions on radial and angular components

We want to define a complex distribution, which in polar form looks like some kind of normal distribution, along both of its radial and angular marginal distributions.

We can try to define it first as follow : $Z = Re^{i\Theta}$, where $R \sim \mathcal{N}(\mu_R, \sigma_R^2)$ and $\Theta \sim \mathcal{N}(\mu_\Theta, \sigma_\Theta^2)$.

However, we already face several issues with this definition : even though it is well defined,

Thus we want R and Θ such that they really represent the modulus and argument of the complex variable, i.e. we should have :

$$|Z| = |R| = R$$

and

$$\arg Z = \Theta$$

where $\arg z \in]-\pi, \pi]$

It confirms the seemingly reasonable assumption that R should be defined only for positive real numbers, and Θ on $]-\pi,\pi]$. Moreover, as it represents a circular distribution, it should satisfy $f_\Theta(-\pi^+) = f_\Theta(\pi)$ to ensure continuity of the angles on a circular representation.

The most suitable candidates for such distributions are :

- A **folded normal distribution** for R, which is simply the distribution of the absolute value of a normal distribution
- A **wrapped normal distribution**, which is, as the name suggests the "wrapping" of a normal distribution along the unit circle

However, the wrapped normal distribution is not tractable, but is fortunately well approximated by either the von Mises distribution or a normal distribution.

See :

https://en.wikipedia.org/wiki/Folded_normal_distribution
https://en.wikipedia.org/wiki/Circular_distribution
https://en.wikipedia.org/wiki/Wrapped_normal_distribution
https://en.wikipedia.org/wiki/Von_Mises_distribution

4 Transformations of complex random variables

In this section, let X and Y be complex random variables on $\mathbf{C}$, with probability distribution functions f_X, f_Y.

4.1 Addition

Let $Z = X + Y$, then :

$$f_Z(z) = (f_X * f_Y)(z) = \iint_{\mathbf{C}} f_X(t) f_Y(z-t) dt \tag{1}$$

4.2 Product

4.2.1 Idea

Let $Z = XY$

The distribution of the product of two complex random variables can be found by computing the marginal distributions in polar form, then computing the product distribution in polar form as well :

If $X = R_X e^{i\Theta_X}$ and $X = R_Y e^{i\Theta_Y}$, then $Z = XY = R_X R_Y e^{\Theta_X + \Theta_Y}$.

We already know how to compute the distribution of the sum of random variables using convolution, and the product can be computed using an integral (or a logarithmic integral!)

https://en.wikipedia.org/wiki/Distribution_of_the_product_of_two_random_variables
https://en.wikipedia.org/wiki/Algebra_of_random_variables

However this method would require a lot of conversions between Cartesian and polar coordinates, I haven't tested if it is worth yet.

Instead, we can use the following identity :

$$Z = XY = e^{\log X + \log Y}$$

Where log is the principal branch of the complex logarithm
https://en.wikipedia.org/wiki/Complex_logarithm.

Note that even if the above identity holds, the following one doesn't always $\log XY = \log X + \log Y$, which can differ by a multiple of $2\pi i$.

Therefore, if we can compute the exponential and logarithm distribution of a complex random variable, we can compute the product of any two complex random variables.

The proof of the following identities are detailed in the appendix, and relate to the multivariate transformation method, see Theorem 2.14 of http://parker.ad.siu.edu/Olive/ich2.pdf.

4.2.2 Complex logarithm

Let $Z = \log X$, we consider only one branch of the logarithm, i.e. $Z = \log X = \log |X| + j \arg X$ with $\arg X \in]-\pi, \pi]$. This means $f_Z = f_{\log X}$ is only defined on the band $]-\infty, \infty[+ j] - \pi, \pi]$. Then we have :

$$f_Z(z) = f_X(e^z)|e^z| = f_X(e^z) e^{\Re(z)} \tag{2}$$

4.2.3 Complex exponential

Let $Z = e^X$.

As opposed to the complex logarithm, which defines a bijective mapping between the complex plane to the previously defined band, the exponential maps each horizontal band of width $2\pi i$ to the entire complex plane. This means that the equation $e^z = x$ as an infinite number of solutions, all of the form $z = \log x + 2k\pi i$, for $k \in \mathbf{Z}$.

This translates to the formula for the distribution of the exponential, where all solutions need to be "folded" in order to obtain the result :

$$f_Z(z) = \sum_{k \in \mathbf{Z}} \frac{f_X(\log z + 2k\pi j)}{|z|} \tag{3}$$

However, we'll see that in our case, to compute the product distribution, it won't be necessary to compute this sum as it is.

4.2.4 Product probability distribution

With these tools in hands, we can finally compute the actual product distribution for the complex random variable :

$$f_Z(z) = \sum_{k \in \mathbf{Z}} \frac{f_{\log X + \log Y}(\log z + 2k\pi j)}{|z|}$$

However, notice that the support of $f_{\log X + \log Y}$ is actually included in $]-\infty, \infty[+j]-2\pi, 2\pi]$, which means that the summand is non-zero only for $k \in \{-1, 0, 1\}$. Thus we can limit the sum to these three values only :

$$f_Z(z) = \sum_{k \in \{-1,0,1\}} \frac{f_{\log X + \log Y}(\log z + 2k\pi j)}{|z|} \tag{4}$$

Thus giving a way to compute the product distribution, by first computing the logartithm distributions, then summing them by convolution, and finally using this last formula for the result.

4.2.5 Link with logarithmic convolution

$$\begin{aligned} f_Z(z) &= \sum_{k \in \{-1,0,1\}} \frac{(f_{\log X} * f_{\log Y})(\log z + 2k\pi j)}{|z|} \\ &= \sum_{k \in \{-1,0,1\}} \frac{1}{|z|} \iint_{|\Im(t)|<\pi} f_{\log X}(\log z + 2k\pi j - t) f_{\log Y}(t) dt \end{aligned}$$

Needs a check on how to split the sum properly, however no matter how it cancels out anyway after and we have :

$$\begin{aligned} f_Z(z) &= \frac{1}{|z|} \iint_{|\Im(t)|<\pi} f_X(e^{\log z + 2k\pi j - t}) |e^{\log z + 2k\pi j - t}| f_Y(e^t) |e^t| dt \\ &= \frac{1}{|z|} \iint_{|\Im(t)|<\pi} f_X(ze^{-t}) |ze^{-t}| f_Y(e^t) |e^t| dt \\ &= \iint_{|\Im(t)|<\pi} f_X(ze^{-t}) f_Y(e^t) dt \\ &= \iint_{|\Im(t)|<\pi} s_X(\log z - t) s_Y(t) dt \\ &= (s_X * s_Y)(\log z) \\ &= (f_X *_l f_Y)(z) \end{aligned} \tag{5}$$

where $s_X(t) = f_X(e^t)$ and $s_Y(t) = f_Y(e^t)$. This is in fact the logarithmic convolution of f_X and f_Y.

See https://en.wikipedia.org/wiki/Logarithmic_convolution.

This is interesting to see that it could provide an efficient way to compute the product distribution for positive real random variables. Otherwise, a formula is known of the product distribution of two real random variables X and Y :

$$f_{XY}(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y(\frac{z}{x}) dx \tag{6}$$

https://en.wikipedia.org/wiki/Distribution_of_the_product_of_two_random_variables

4.3 Squared magnitude

For the following sections, let X and Y be real random variables. We are interested in the real random variable $Z = |X + iY|^2 = X^2 + Y^2$.

We present two ways to compute the distribution of Z, the squared magnitude of the complex random variable with real part and imaginary part X and Y. The first way involves directly computing the cumulative distribution function, the second one by first computing the distribution of the squares, then summing them.

4.3.1 Square

The distribution of the square of a positive real random variable is given by :

$$f_{X^2}(x) = \frac{f_X(\sqrt{x})}{2\sqrt{x}} \tag{7}$$

If the random variable is defined for all real numbers, then the distribution is :

$$f_{X^2}(x) = \frac{f_X(\sqrt{x}) + f_X(-\sqrt{x})}{2\sqrt{x}} \tag{8}$$

4.3.2 CDF : direct computation

$$\begin{aligned} F_Z(z) &= \int_{-\sqrt{z}}^{\sqrt{z}} f_Y(y) \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_X(x) dx dy \\ &= \int_{-\sqrt{z}}^{\sqrt{z}} f_Y(y)(F_X(\sqrt{z-y^2}) - F_X(-\sqrt{z-y^2}))dy \end{aligned} \tag{9}$$

Thus the pdf is :

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_{-\sqrt{z}}^{\sqrt{z}} f_Y(y)(F_X(\sqrt{z-y^2}) - F_X(-\sqrt{z-y^2}))dy \tag{10}$$

4.3.3 As a sum of squares

$$\begin{aligned} f_Z(z) &= f_{X^2+Y^2}(z) \\ &= (f_{X^2} * f_{Y^2})(z) \\ &= \int_0^z f_{X^2}(z-t) f_{Y^2}(t) dt \\ &= \int_0^{\sqrt{z/2}} \frac{(f_X(u) + f_X(-u))(f_Y(\sqrt{z-u^2}) + f_Y(-\sqrt{z-u^2}))}{\sqrt{z-u^2}} \\ &+ \frac{(f_Y(u) + f_Y(-u))(f_X(\sqrt{z-u^2}) + f_X(-\sqrt{z-u^2}))}{\sqrt{z-u^2}} du \end{aligned} \tag{11}$$

Computing the convolution directly would be the fastest, but it's not stable due to the potential singularity at 0 for the distribution of the squares. To compensate, we compute the last line of the formula, which is obtained after a change of variable to avoid the singularities.

A Proofs