# Complex Interval Arithmetic for ultrasound imaging

Surveying subsea cables and pipelines with non-ideal sonar

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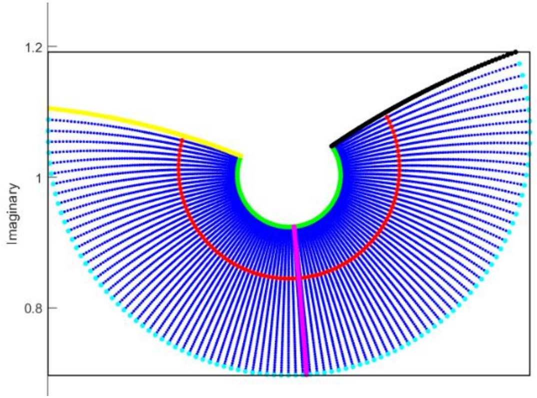
Overview of summer project and M1 internship in the DSB group, supervised by Gábor Geréb[2]

[1]ENS Rennes, Computer Science department [2]DSB group, Department of Informatics (IFI)

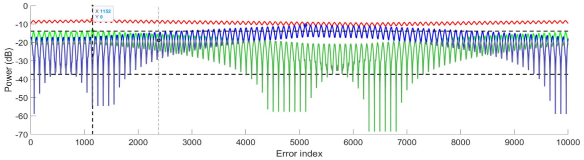
# Introduction

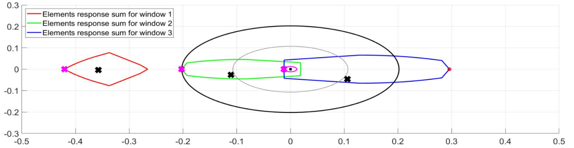
Part of summer project




Study of a tool for the worst-case analysis of non-ideal sonars

Sensitivity of adaptive array signal processing to calibration errors



# Table of contents

Three main parts :

# Complex Interval Arithmetic

How to reason with uncertainty

- Real Interval Arithmetic
- Complex Interval Arithmetic
- Interval extensions of functions
- Dependency problem
- MATLAB Toolbox

# Real interval arithmetic

- Compute with intervals instead of numbers
- Representation of uncertainty, and worst-case scenario, without needing statistics
- Simple on the real line : only one way to define it (connected closed sets)

<u>Example :</u>

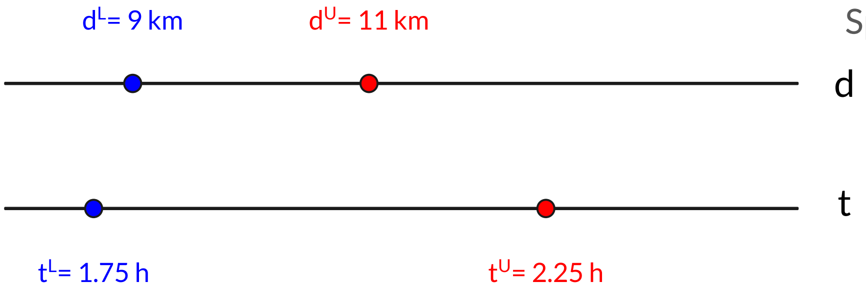
Traveled ≈ 10 km in ≈ 2 hours → Speed ≈ 5 km/h

<u>With intervals :</u>

Distance : $d^I = [d^L, d^U] = [9, 11]$ km

Time : $t^I = [t^L, t^U] = [1.75, 2.25]$ h

Speed : $v^I = d^I / t^I = [d^L/t^U, d^U/t^L] = [4, 6.29]$ km/h



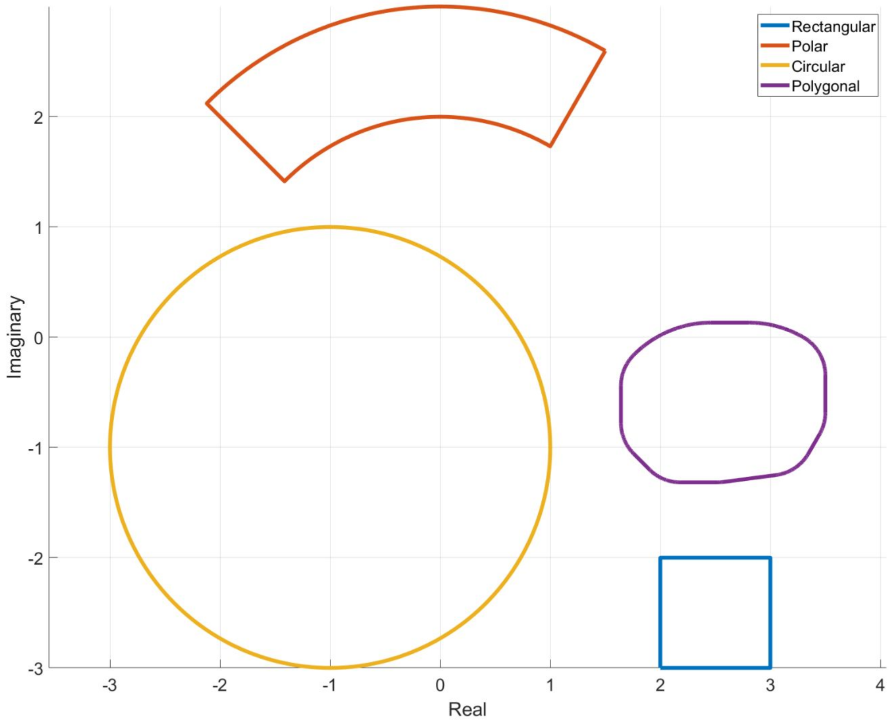

# Extension to complex numbers

Several ways to define complex “intervals”

- *Rectangular*
- *Polar*
- *Circular*
- *Polygonal*
- …

Each have advantages and drawbacks :

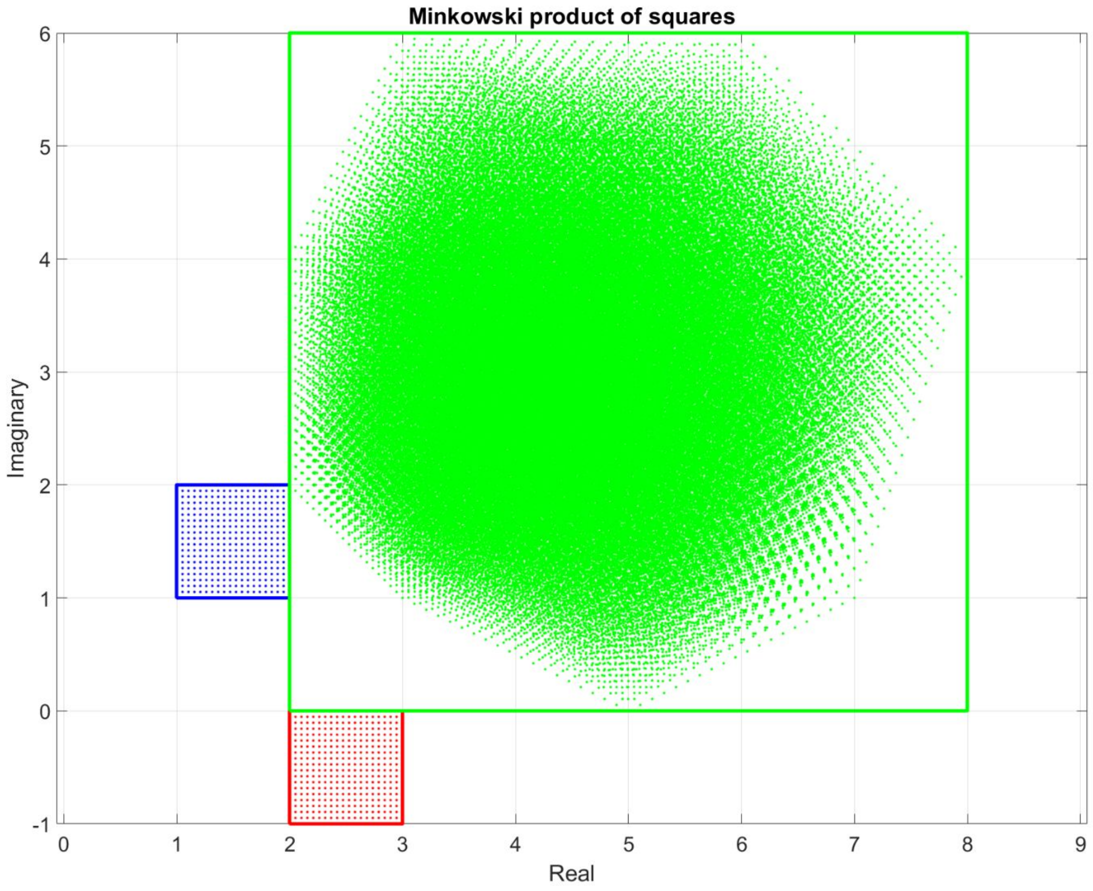
- Computational cost
- Tightness
- Closure under some operations
- …



# Operations on complex intervals

Minkowski (pointwise) product of squares forms an octagon.

The IA extension returns a tight rectangle containing it (i.e. the rectangular interval hull of the image).



# Dependency problem

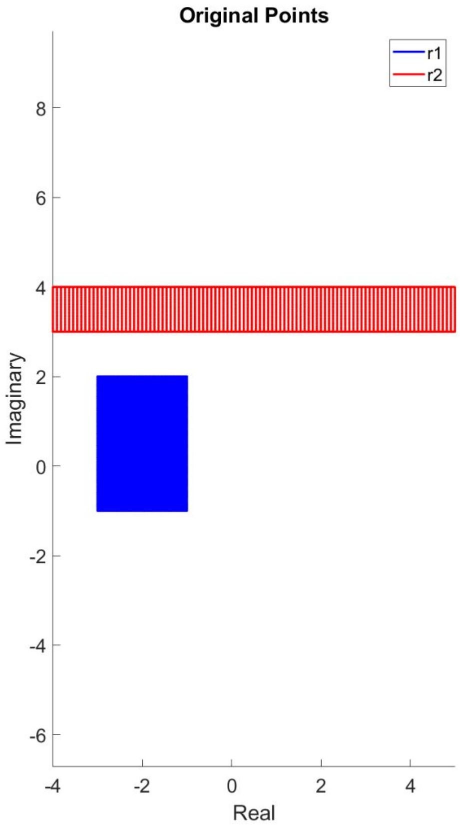
Occurs when a variable appears several time in an expression

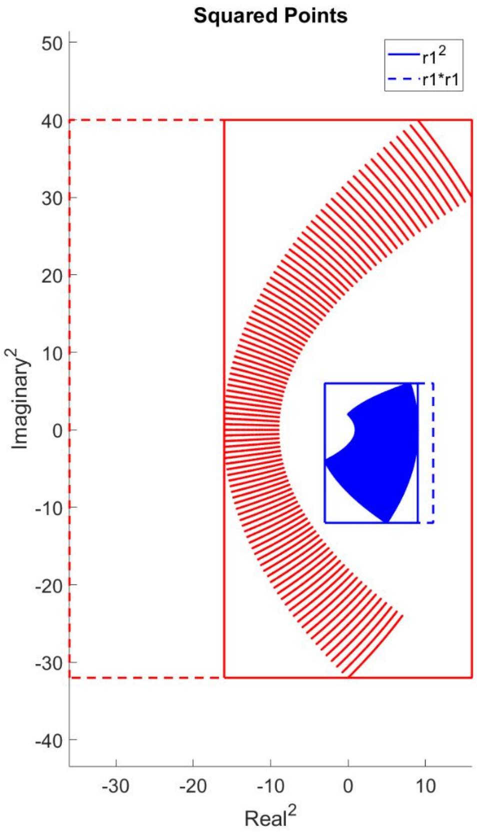
<u>Example :</u>

$x^I = [-1, 1]$ ; Square of $x^I$ ?

$x^I * x^I = [-1, 1] * [-1, 1] = [-1, 1]$

$(x^I)^2 = [0, 1] \neq [-1, 1]$



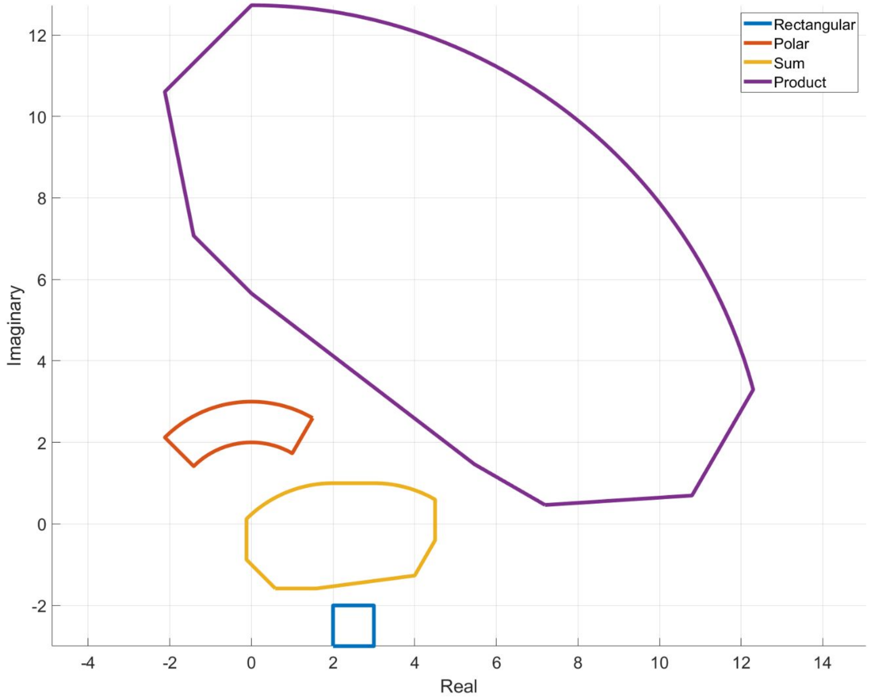



# Complex Interval Arithmetic Toolbox (CIAT)

MATLAB Toolbox (Gábor Geréb - Håvard Kjellmo Arnestad)

Contributions :

- Improvements and bugfixes
- Interval extensions of many functions
- IA with probability (see last section)




Havard Kjellmo Arnestad, Gábor Geréb, et al. ; Sonar array beampattern bounds and an interval arithmetic toolbox. Proc. Mtgs. Acoust 2022

# Application to sonar arrays

Modelization of non-ideal sonar

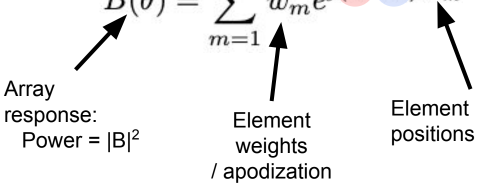
- Sonar beamforming
- An adaptive beamformer LCA
- How to use IA
- Can we have tighter bounds?

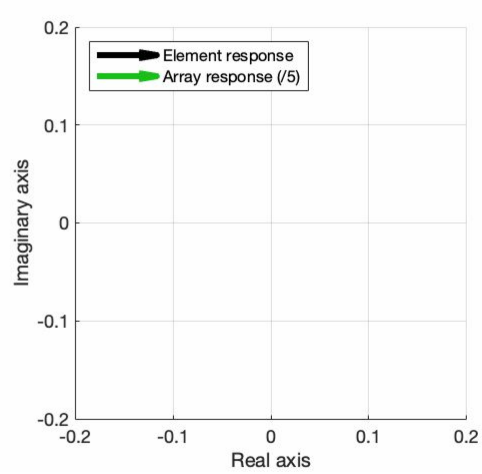
*Slide by Håvard Arnestad[1], ICUA 2022*

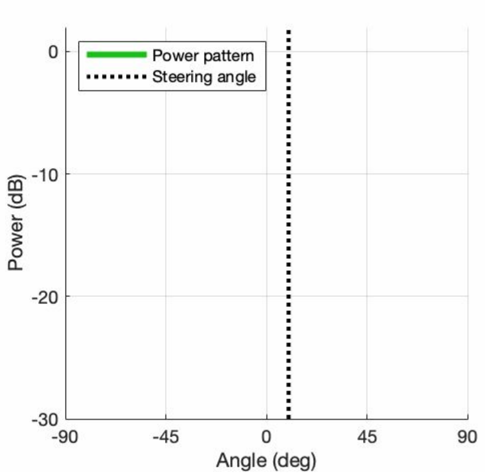
# Sonar beamforming

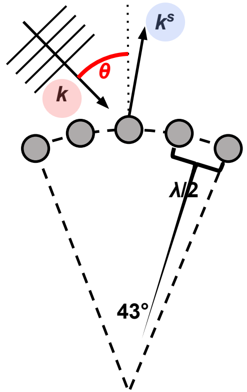
A summation of element responses to enhance pick-up from a particular location/direction

$$B(\theta) = \sum_{m=1}^{M} w_m e^{j(\boldsymbol{k}-\boldsymbol{k}^s)\cdot \boldsymbol{x}_m}$$




**Curved *M* = 5 element array**

**Sonar rarely ideal!**
- Component aging
- Temperature variations
- Environmental factors (e.g., barnacle)
- Manufacturing imperfections
- Damages and deformations
- Design flaws
- Model errors

**Manifests as errors in:**
- Amplitude
  - Element response
  - Orientation error (directivity)
- Phase
  - Element response
  - Position errors
- Coupling

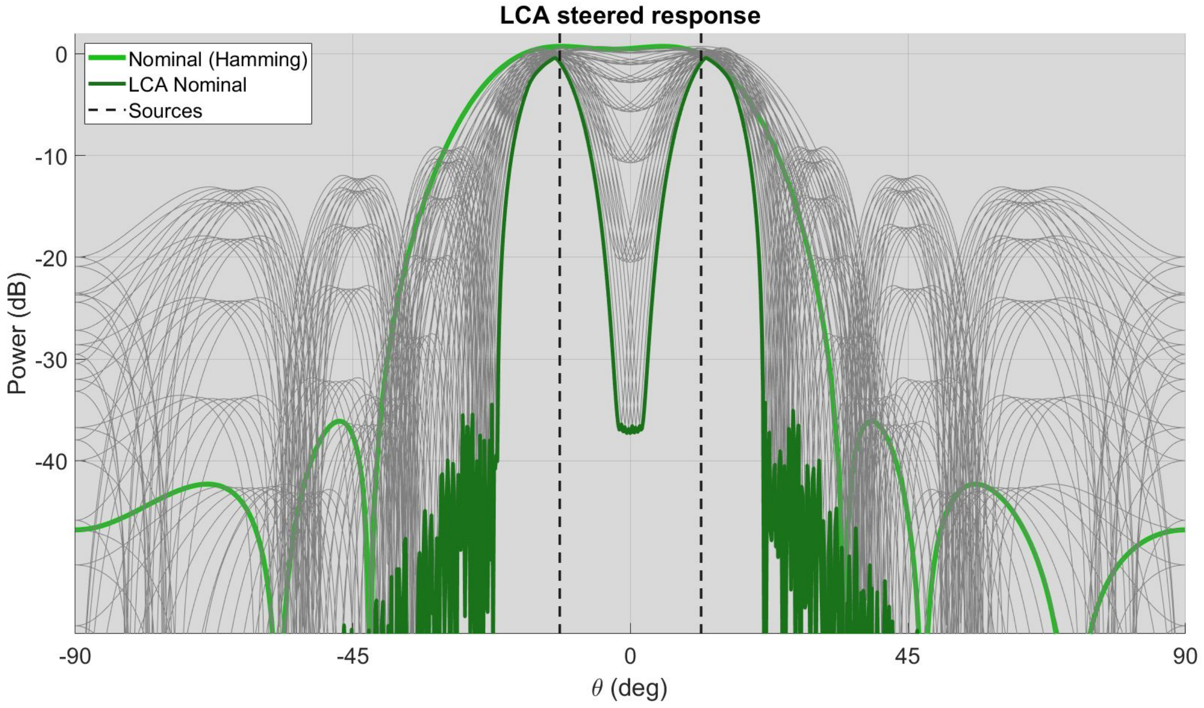


[1]DSB group, Department of Informatics (IFI)

# Low Complexity Adaptive Beamformer (LCA)

- Discrete version of MVDR
- Adaptive version of DAS

Adaptive beamformers seek to reduce the power of noise and interference.

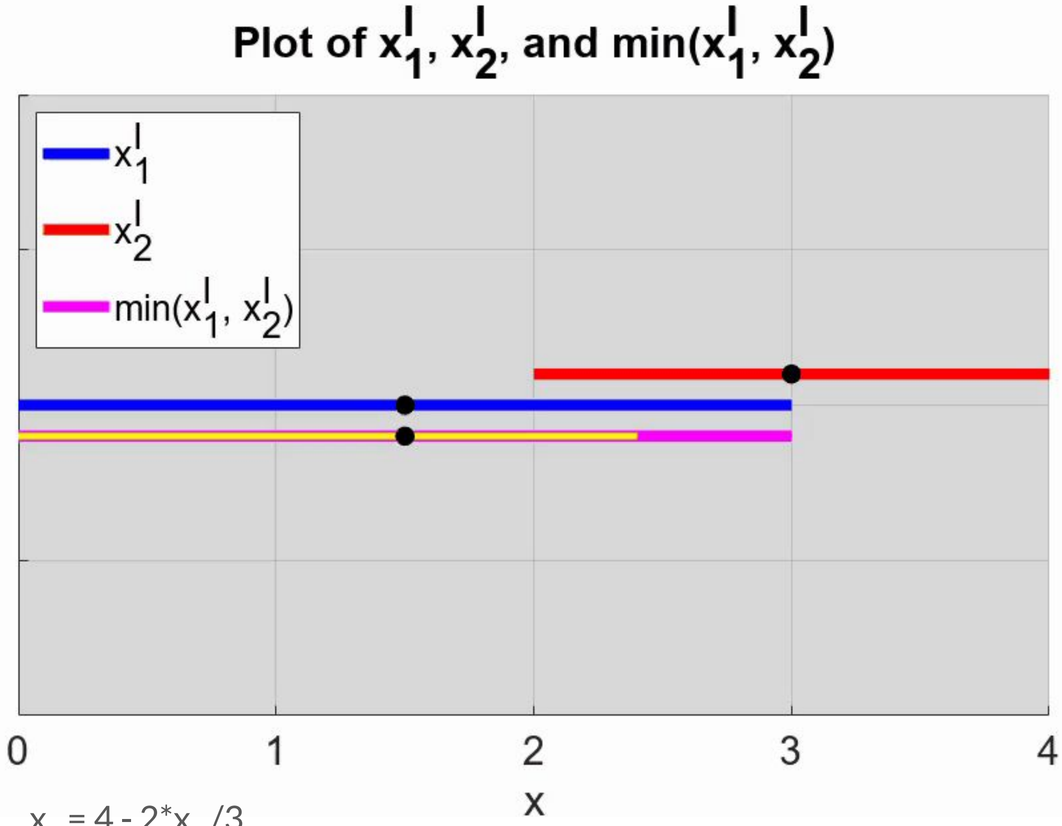



J. I. Buskenes et al., "Low-Complexity Adaptive Sonar Imaging," in IEEE Journal of Oceanic Engineering 2017.

# Minimum (intervals)

- Independent inputs
- What happens with dependent inputs ?

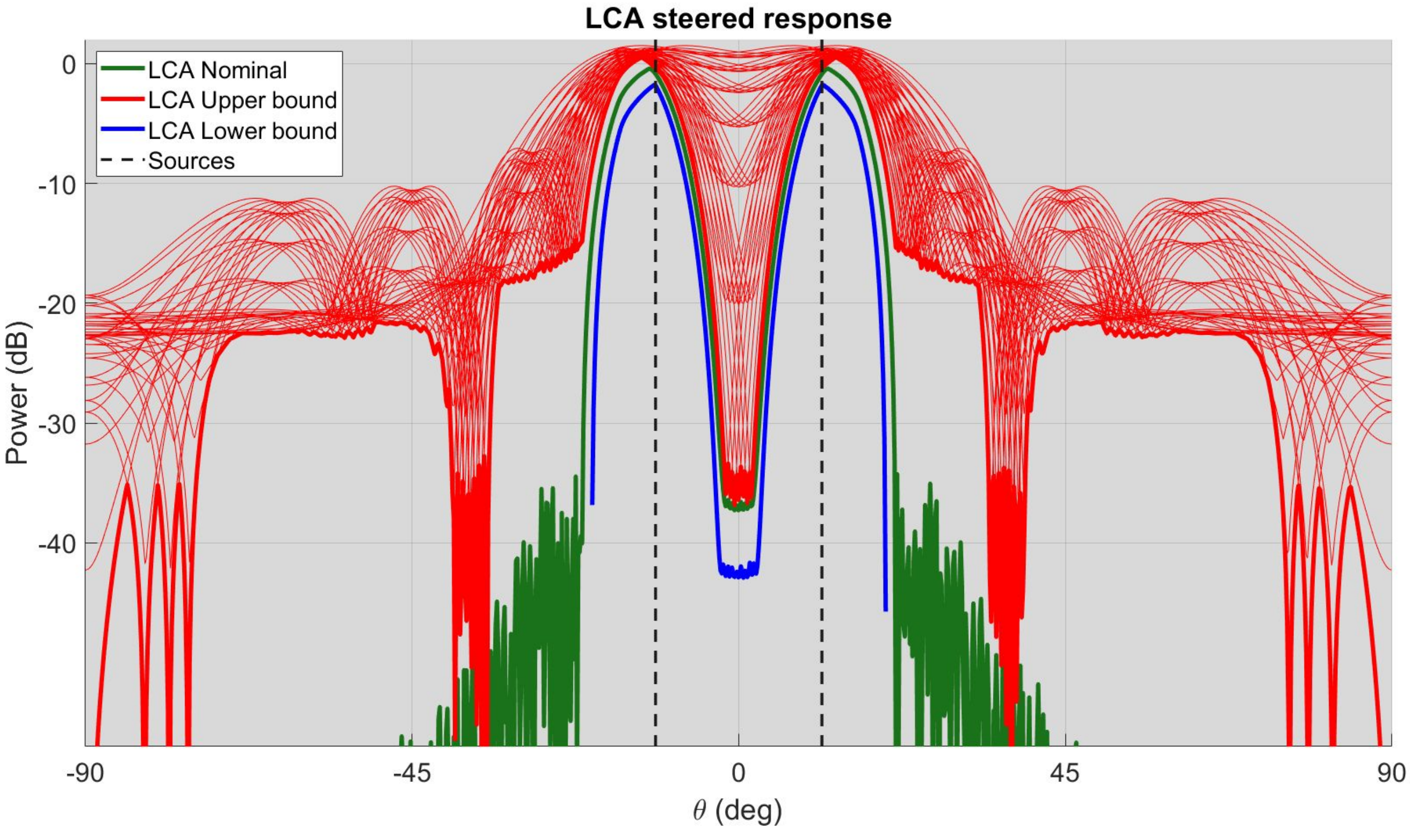
$$\min\{x^I, y^I\} = \{\min\{x, y\} \,|\, \forall x \in x^I\ \forall y \in y^I\}$$
$$= [\min\{\underline{x}^I, \underline{y}^I\}, \min\{\bar{x}^I, \bar{y}^I\}]$$




$x_2 = 4 - 2{*}x_1/3$

# LCA with IA

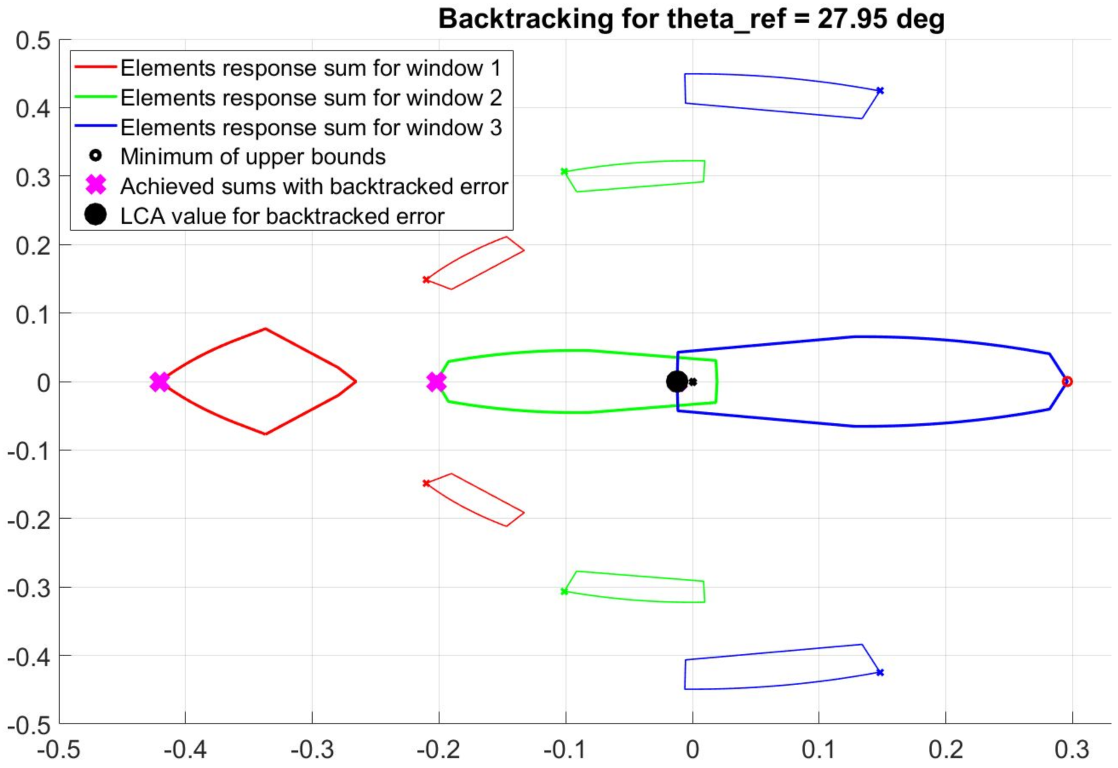
Computes the upper bound for each window, then takes the minimum



# Some other tries

- Using backtracking
- Iterative methods

Hopes to approximate the upper bound

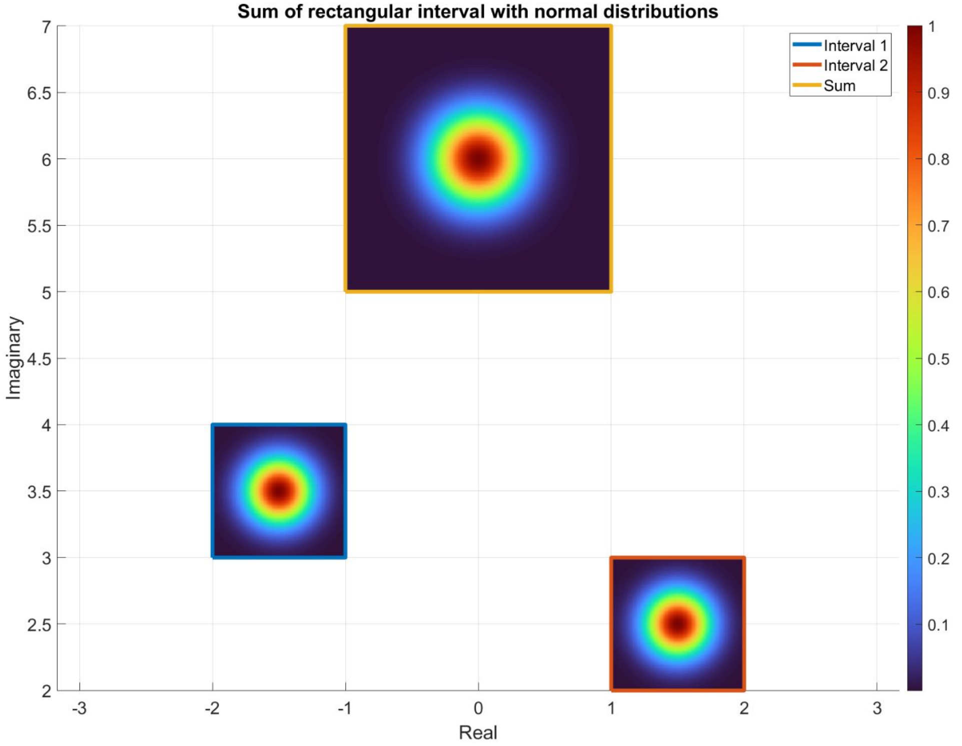


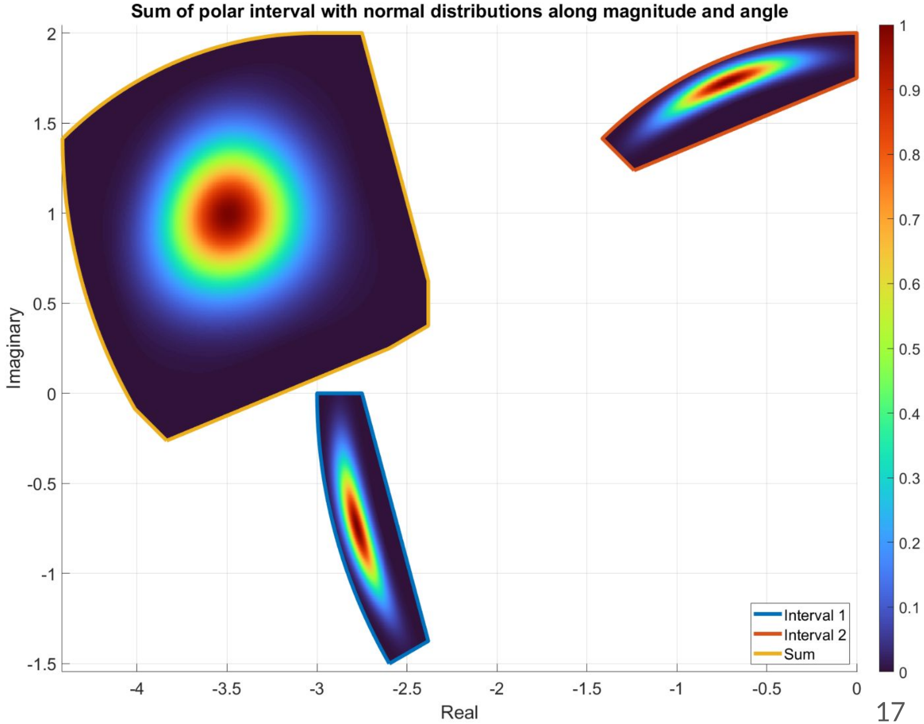
# Probabilities on intervals

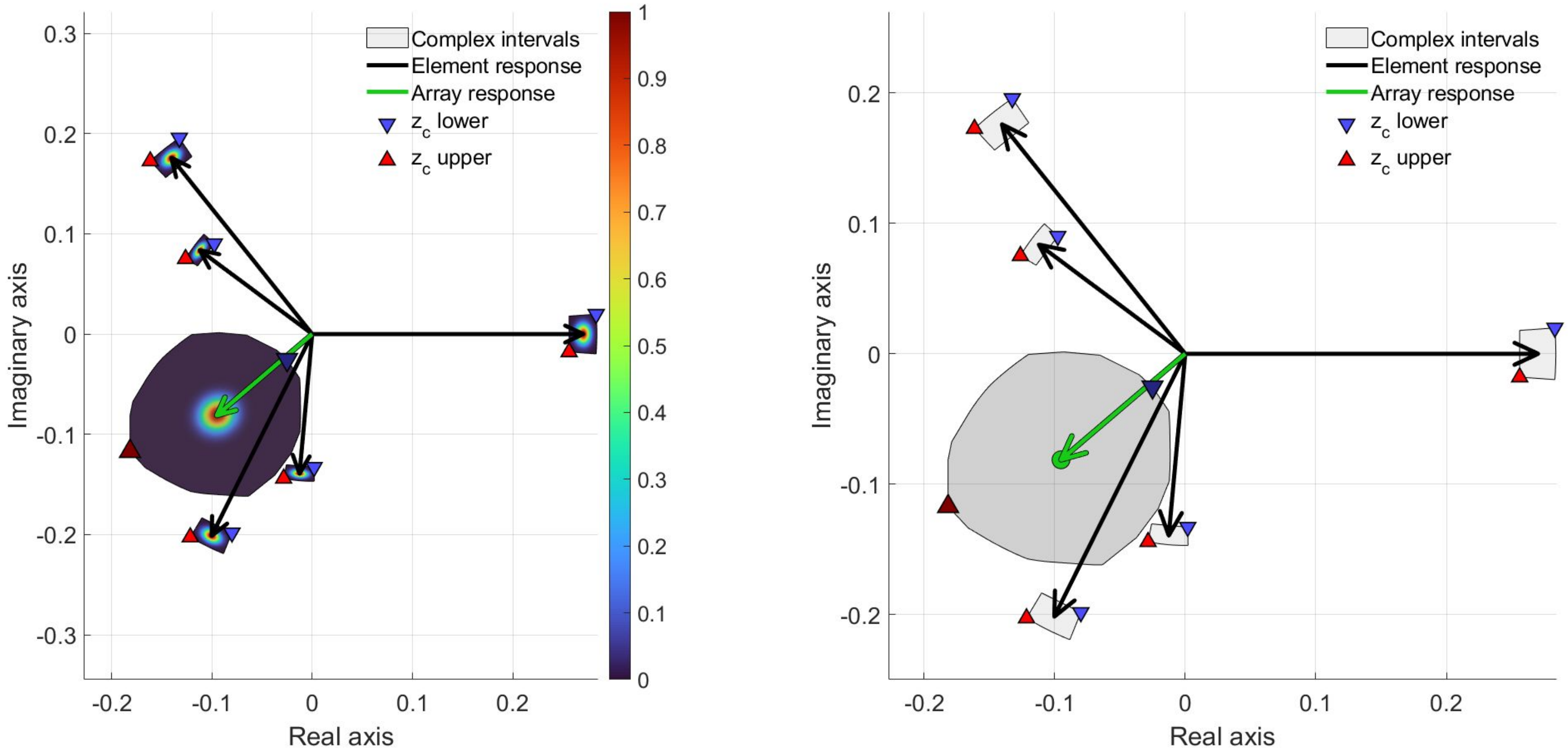
Heatmaps, and back to statistics

- Implementation in the CIAT toolbox
- Operations on random variables
- Usage

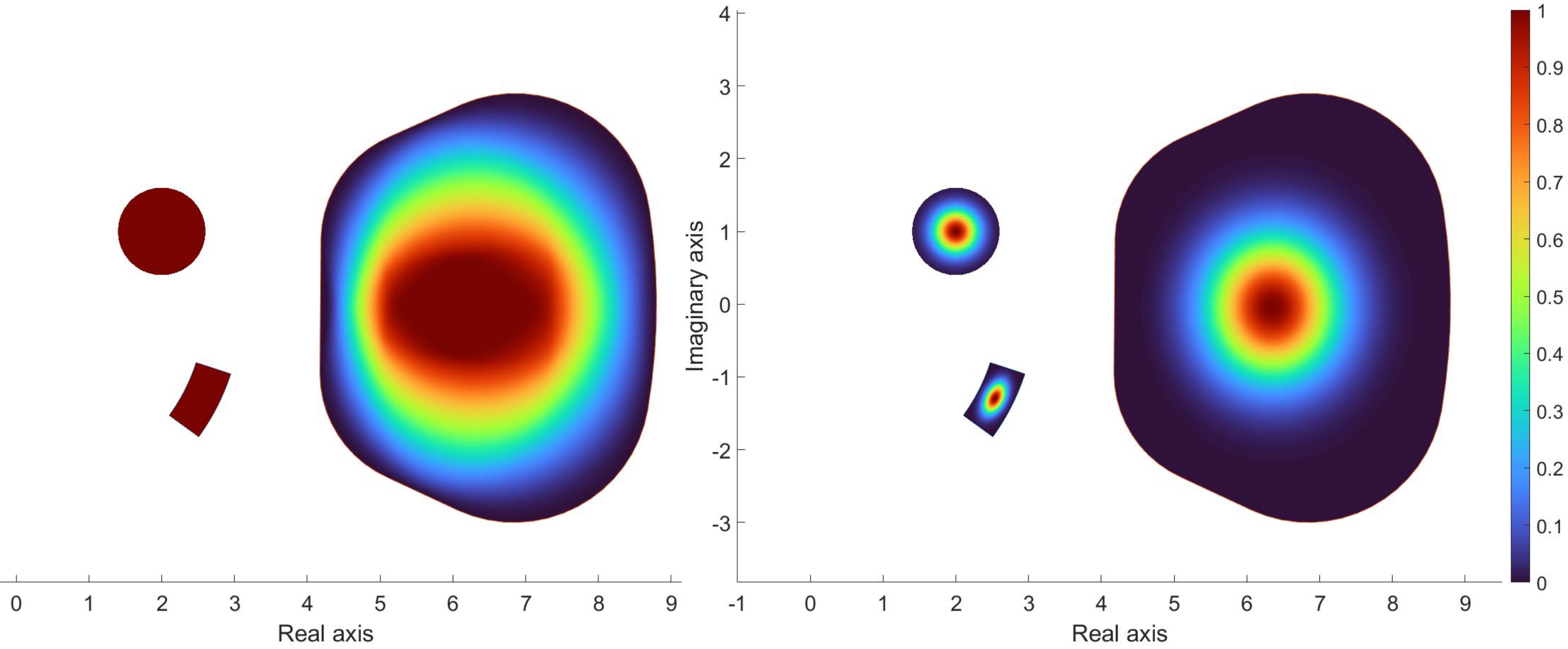
# Two examples



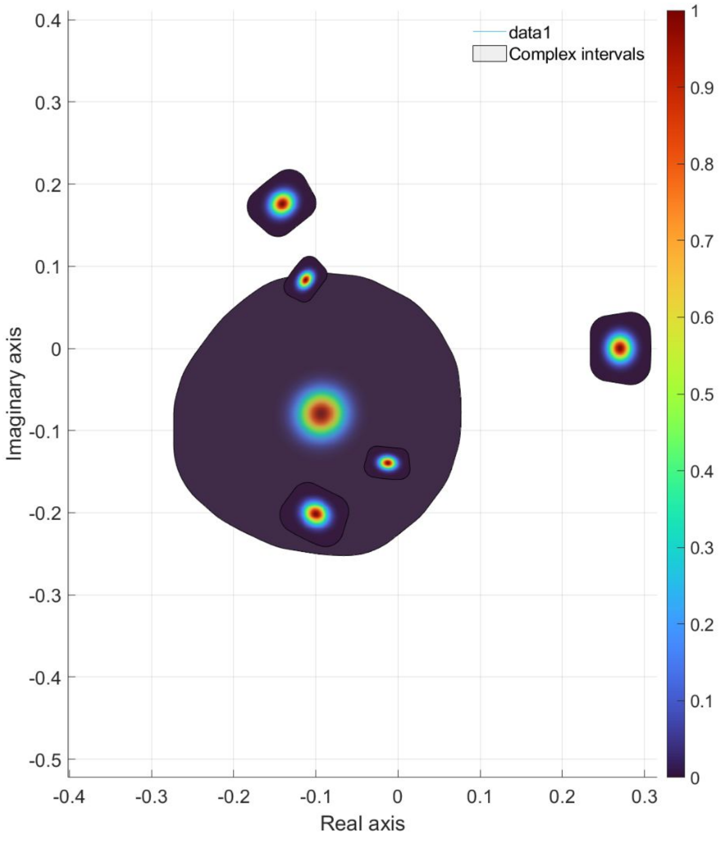


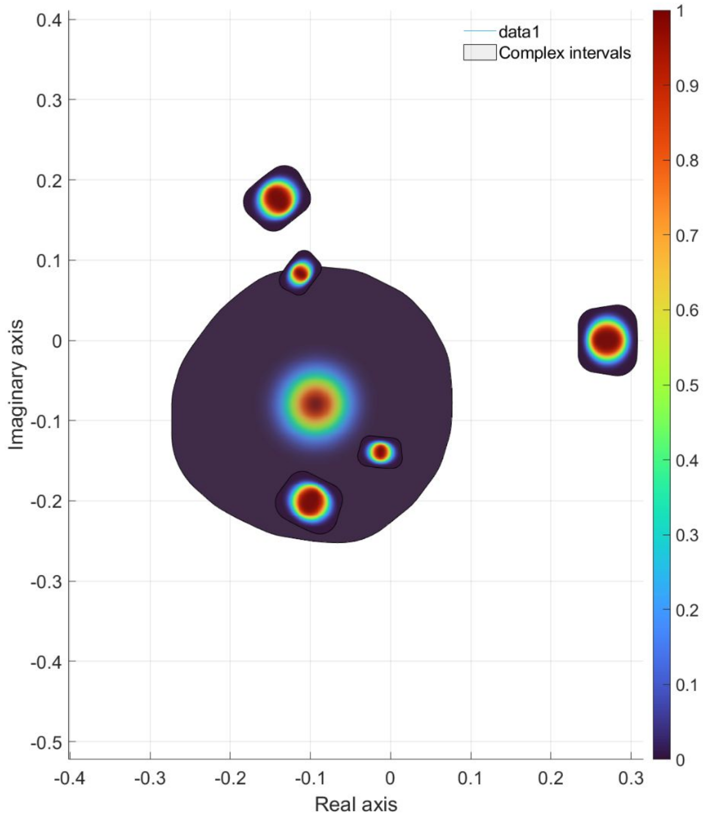

Side by side comparison of Fig. 6. (b) of JASA paper: sum of element response with and without probability distributions




Product of circular and polar intervals (Fig. 8 JASA paper) -

- Left : Product of intervals with uniform distributions
- Right: 2D normal distribution for the circle, normal-ish distribution along magnitude and angle for polar interval

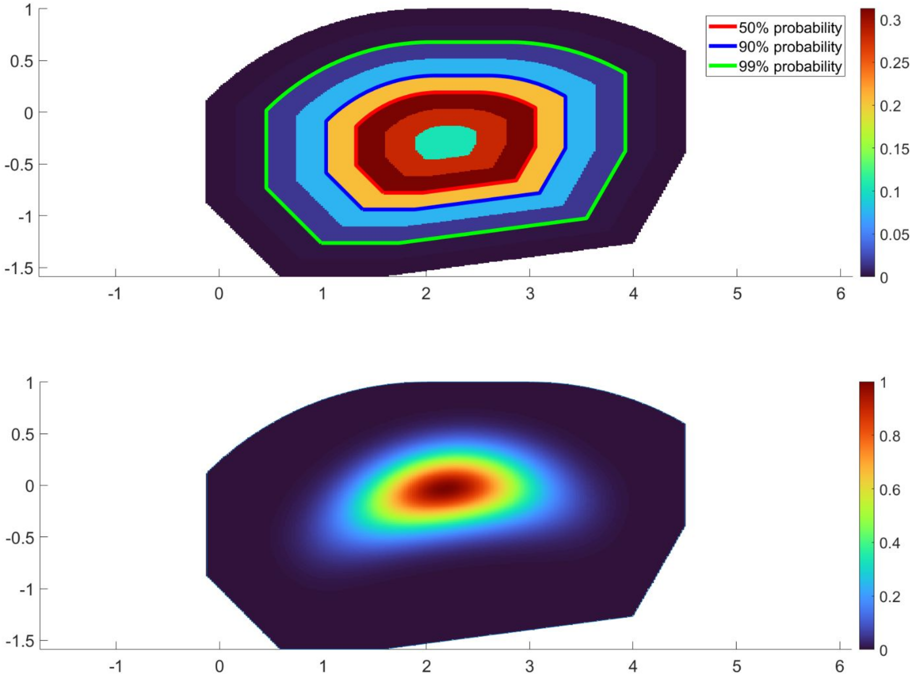


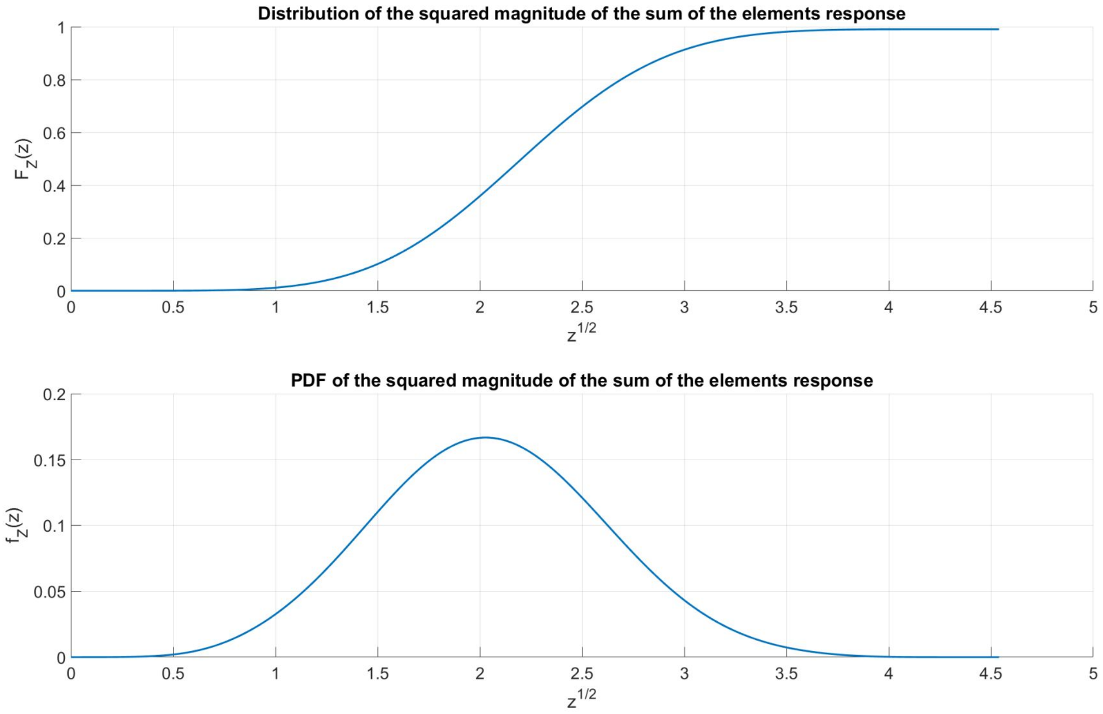

Multiplication of intervals allows a deeper study of coupling errors

# Usage

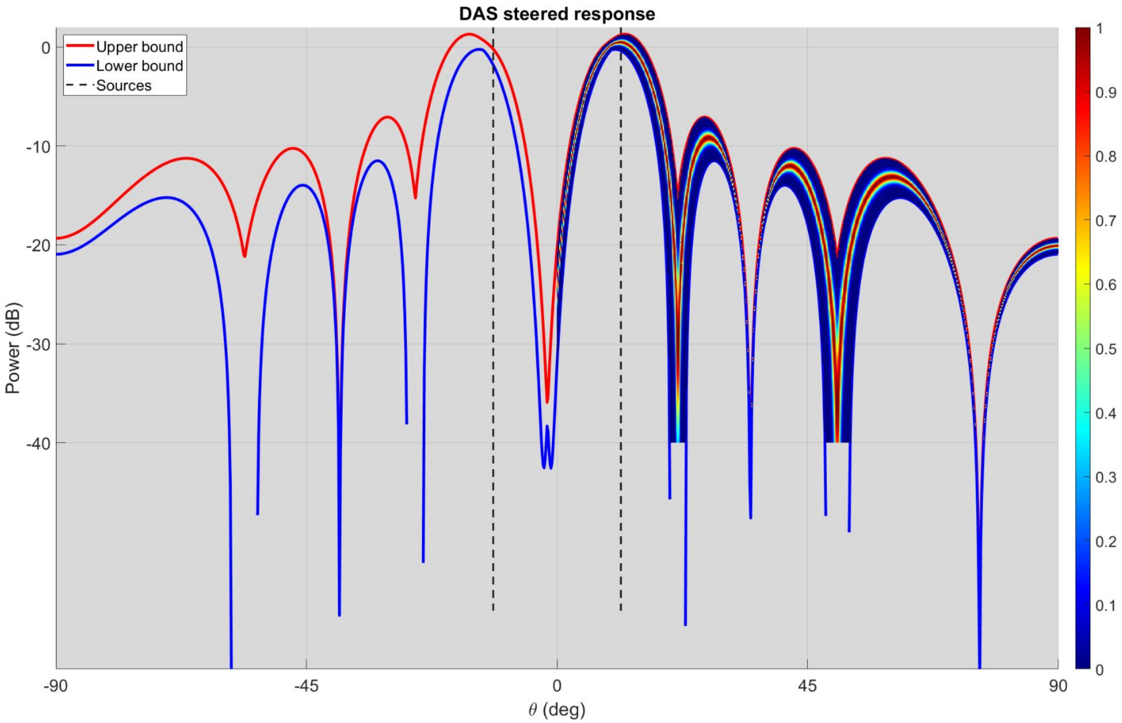
Zone captures 50/90/99 % of the probability



# Power distribution (1)



# Power distribution (2)



# Conclusion

A lot of rectangles…

Done :

- Interval Arithmetic
  - Contribution for the toolbox
- IA for LCA
  - Non-trivial to obtain tight bound
- Probability intervals

→ Two papers

Future works :

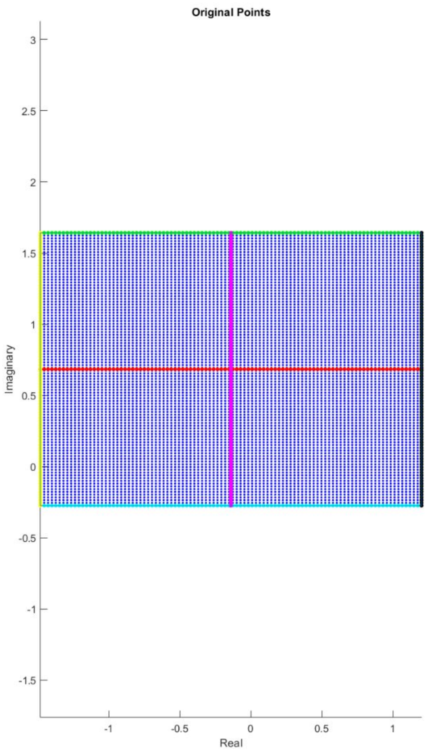
- Deeper analysis using the toolbox
- More insight on adaptive beamformers

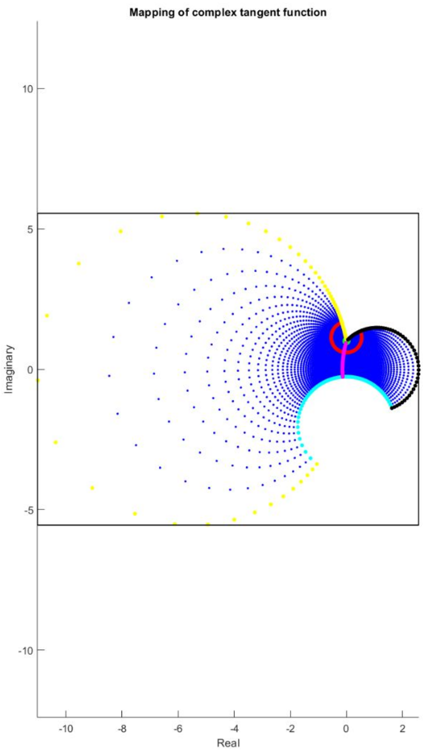
**Thanks for listening!**

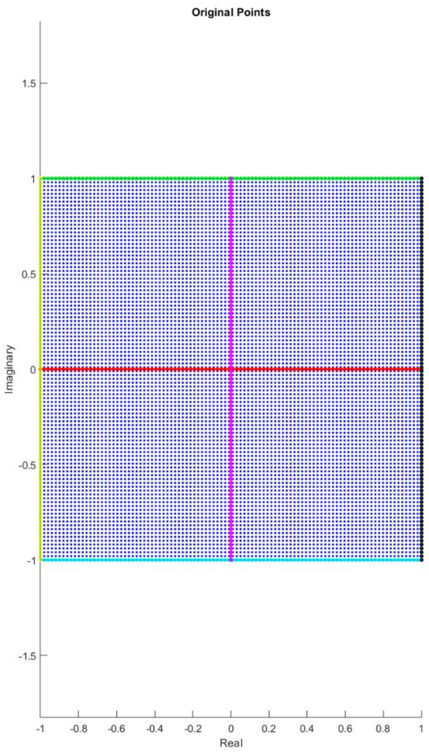
# Appendix

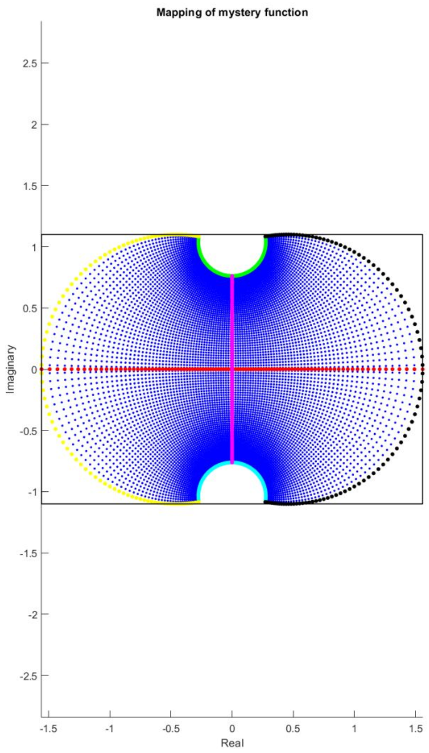
- Additional figures
- Formulas

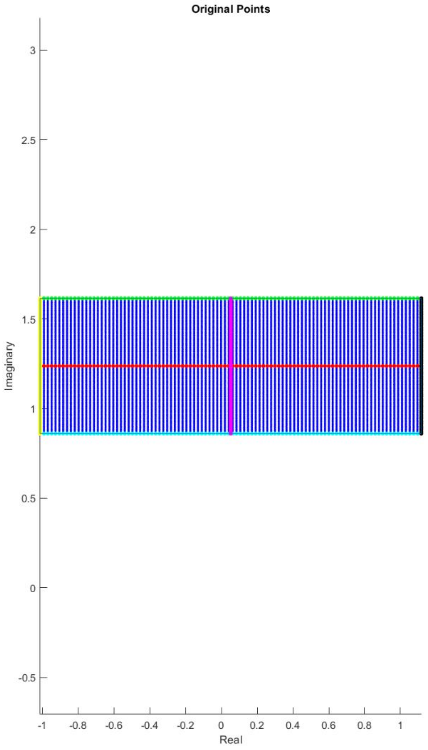
# Interval Analysis

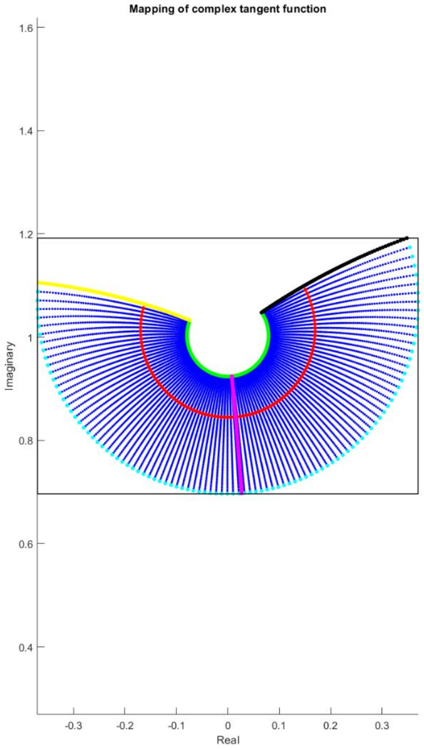

Original Points
Imaginary
Real

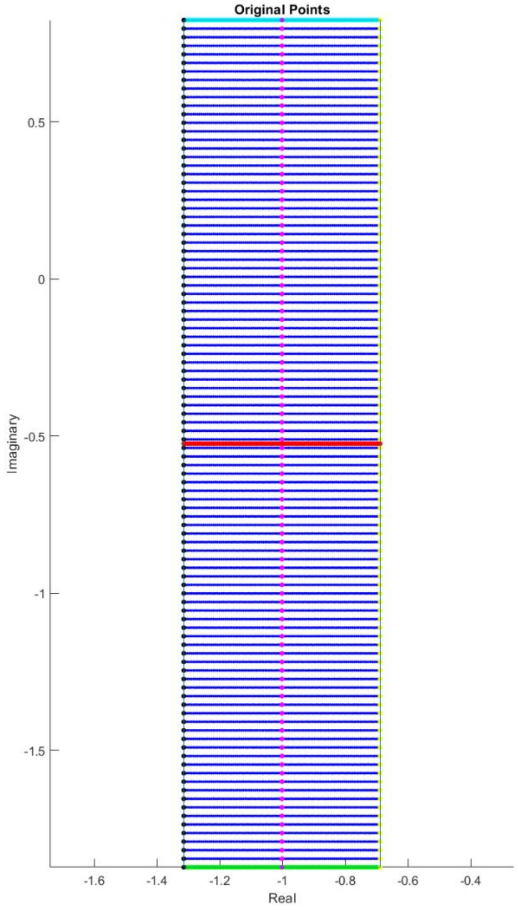

Mapping of complex tangent function
Imaginary
Real


Original Points
Imaginary
Real
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Mapping of mystery function
Imaginary
Real
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1.5
-1
-0.5
0
0.5
1
1.5


Original Points
Imaginary
Real
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Mapping of complex tangent function
Imaginary
Real
1.6
1.4
1.2
1
0.8
0.6
0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3



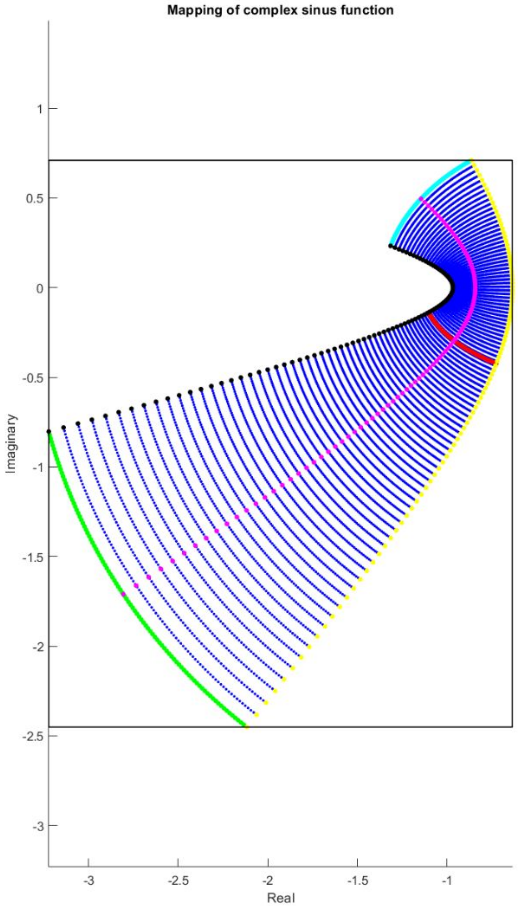
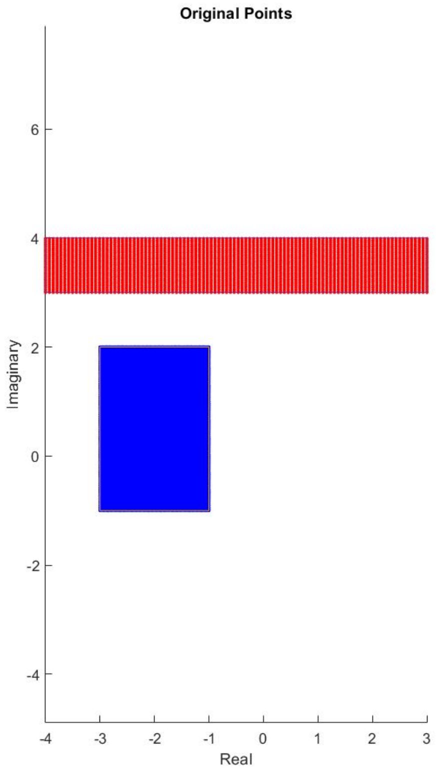
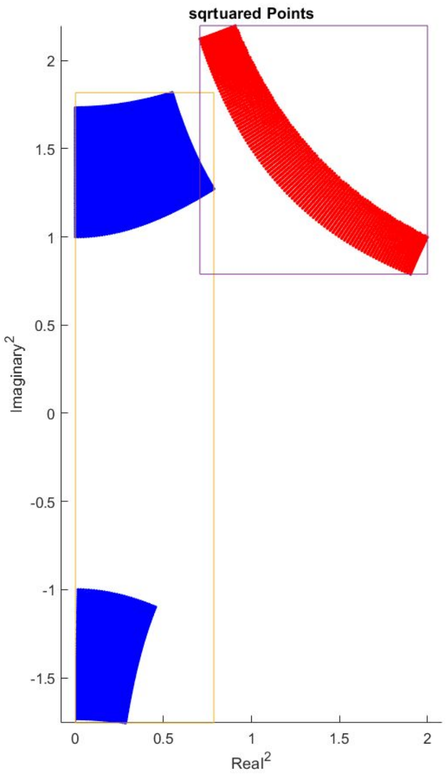




Image of a rectangular interval by the complex sinus function


Original Points
Imaginary
Real
6
4
2
0
-2
-4
-3
-1
1
3


sqrtuared Points
Imaginary$^2$
Real$^2$
2
1.5
1
0.5
0
-0.5
-1
-1.5

# Probabilistic intervals

# Addition

Let $Z = X + Y$, then :

$$f_Z(z) = (f_X * f_Y)(z) = \iint_{\mathbf{C}} f_X(t) f_Y(z-t) dt \qquad (1)$$

# Logarithm

Let $Z = \log X$, we consider only one branch of the logarithm, i.e. $Z = \log X = \log|X| + j \arg X$ with $\arg X \in ]-\pi, \pi]$. This means $f_Z = f_{\log X}$ is only defined on the band $]-\infty, \infty[+j]-\pi, \pi]$. Then we have :

$$f_Z(z) = f_X(e^z)|e^z| = f_X(e^z)e^{\Re(z)}$$

# Exponential

$$f_Z(z) = \sum_{k \in \mathbf{Z}} \frac{f_X(\log z + 2k\pi j)}{|z|}$$

# Product

$$
\begin{aligned}
f_Z(z) &= \sum_{k\in\mathbf{Z}} \frac{(f_{\log X} * f_{\log Y})(\log z + 2k\pi j)}{|z|} \\
&= \sum_{k\in\mathbf{Z}} \frac{1}{|z|} \iint_{|\Im(t)|<\pi} f_{\log X}(\log z + 2k\pi j - t) f_{\log Y}(t) dt \\
&= \frac{1}{|z|} \iint_{|\Im(t)|<\pi} f_X(e^{\log z + 2k\pi j - t}) |e^{\log z + 2k\pi j - t}| f_Y(e^t) |e^t| dt \\
&= \frac{1}{|z|} \iint_{|\Im(t)|<\pi} f_X(ze^{-t}) |ze^{-t}| f_Y(e^t) |e^t| dt \\
&= \iint_{|\Im(t)|<\pi} f_X(ze^{-t}) f_Y(e^t) dt \\
&= \iint_{|\Im(t)|<\pi} s_X(\log z - t) s_Y(t) dt \\
&= (s_X * s_Y)(\log z) \\
&= (f_X *_l f_Y)(z)
\end{aligned}
$$

where $s_X(t) = f_X(e^t)$ and $s_Y(t) = f_Y(e^t)$. This is in fact the logarithmic convolution of $f_X$ and $f_Y$.

# Square

The distribution of the square of a positive real random variable is given by :

$$f_{X^2}(x) = \frac{f_X(\sqrt{x})}{2\sqrt{x}}$$

If the random variable is defined for all real numbers, then the distribution is :

$$f_{X^2}(x) = \frac{f_X(\sqrt{x}) + f_X(-\sqrt{x})}{2\sqrt{x}}$$

# Squared magnitude

Several ways :

- Direct computation of the cdf
- Using formula for the squares and sum using convolution
  - Fast but singularity at zero
  - Stable way to compute the pdf with a change of variable

$$
\begin{aligned}
F_Z(z) &= \int_{-\sqrt{z}}^{\sqrt{z}} f_Y(y) \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_X(x)dxdy \\
&= \int_{-\sqrt{z}}^{\sqrt{z}} f_Y(y)(F_X(\sqrt{z-y^2}) - F_X(-\sqrt{z-y^2}))dy
\end{aligned}
$$

$$
\begin{aligned}
f_Z(z) &= f_{X^2+Y^2}(z) \\
&= (f_{X^2} * f_{Y^2})(z) \\
&= \int_0^z f_{X^2}(z-t)f_{Y^2}(t)dt \\
&= \int_0^{\sqrt{z/2}} \frac{(f_X(u)+f_X(-u))(f_Y(\sqrt{z-u^2})+f_Y(-\sqrt{z-u^2}))+(f_Y(u)+f_Y(-u))(f_X(\sqrt{z-u^2})+f_X(-\sqrt{z-u^2}))}{\sqrt{z-u^2}}du
\end{aligned}
$$

# Distributions

- Uniform
- Complex normal
- Polar normal
  - Wrapped normal distribution & Von Mises distribution
  - Folded normal distribution
  - Truncated normal distribution

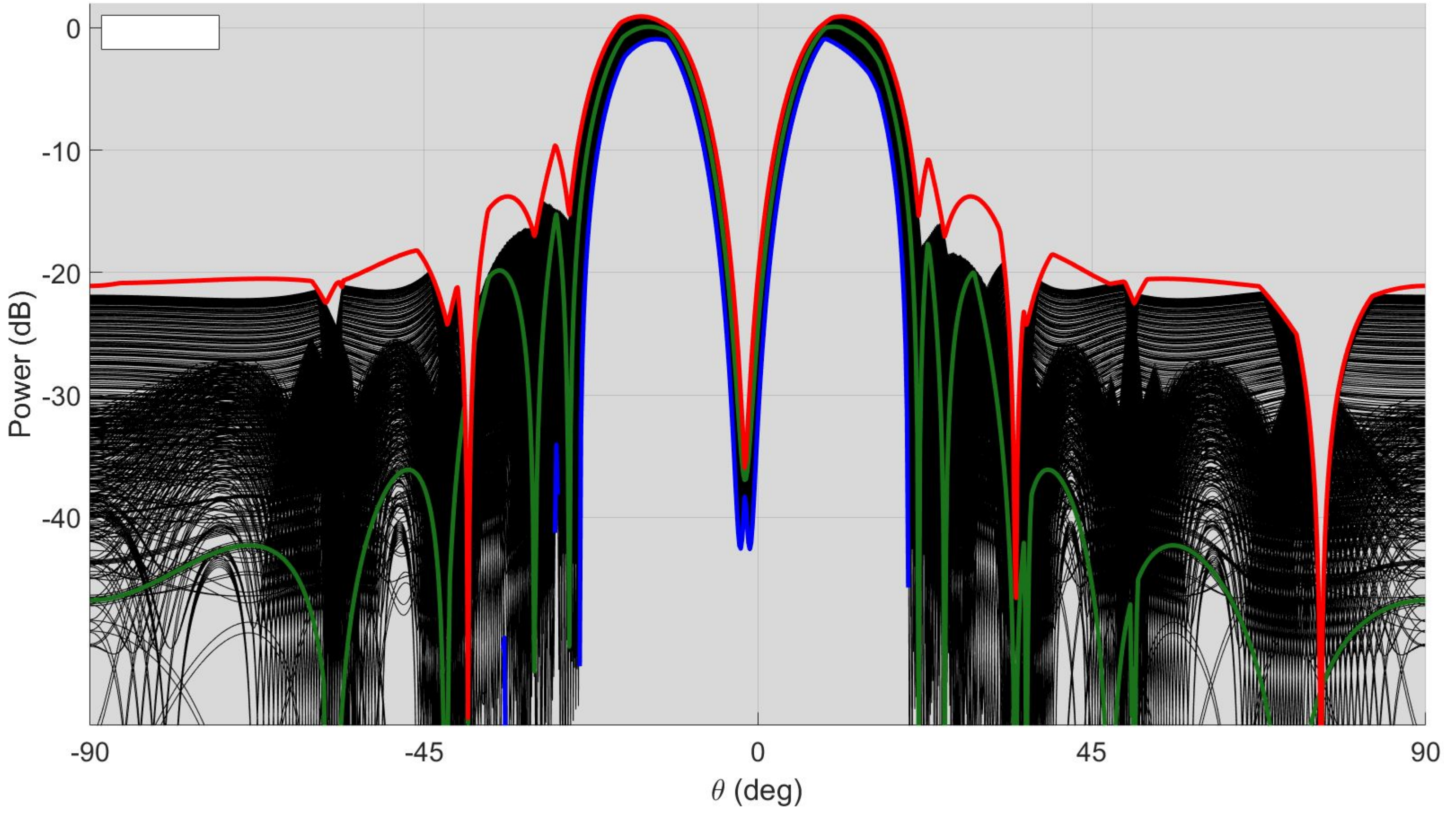
# LCA

# Mathematical formulation

$$P_{LCA}(\theta) = \min_{w \in W} |B_w(\theta)|^2$$

And with intervals?

$$P^I_{LCA}(\theta) = \min_{w \in W} |B^I_w(\theta)|^2$$




Exhaustive Monte-Carlo method for a simple case - Bounds are not tight

# Untight bounds : a dependency problem

Recall the formula for the LCA steered response

$$P^I_{LCA}(\theta) = \min_{w \in W} |B^I_w(\theta)|^2$$

Consider the true (optimal) upper bound :

$$\overline{P}^{opt}_{LCA}(\theta) = \max_{\varepsilon \in E} \min_{\boldsymbol{w} \in W} |B_{\boldsymbol{w}}(\theta, \varepsilon)|^2$$

Compared to the one given by IA

$$\overline{P^I_{LCA}(\theta)} = \min_{\boldsymbol{w} \in W} \overline{|B^I_{\boldsymbol{w}}(\theta_s)|^2}$$

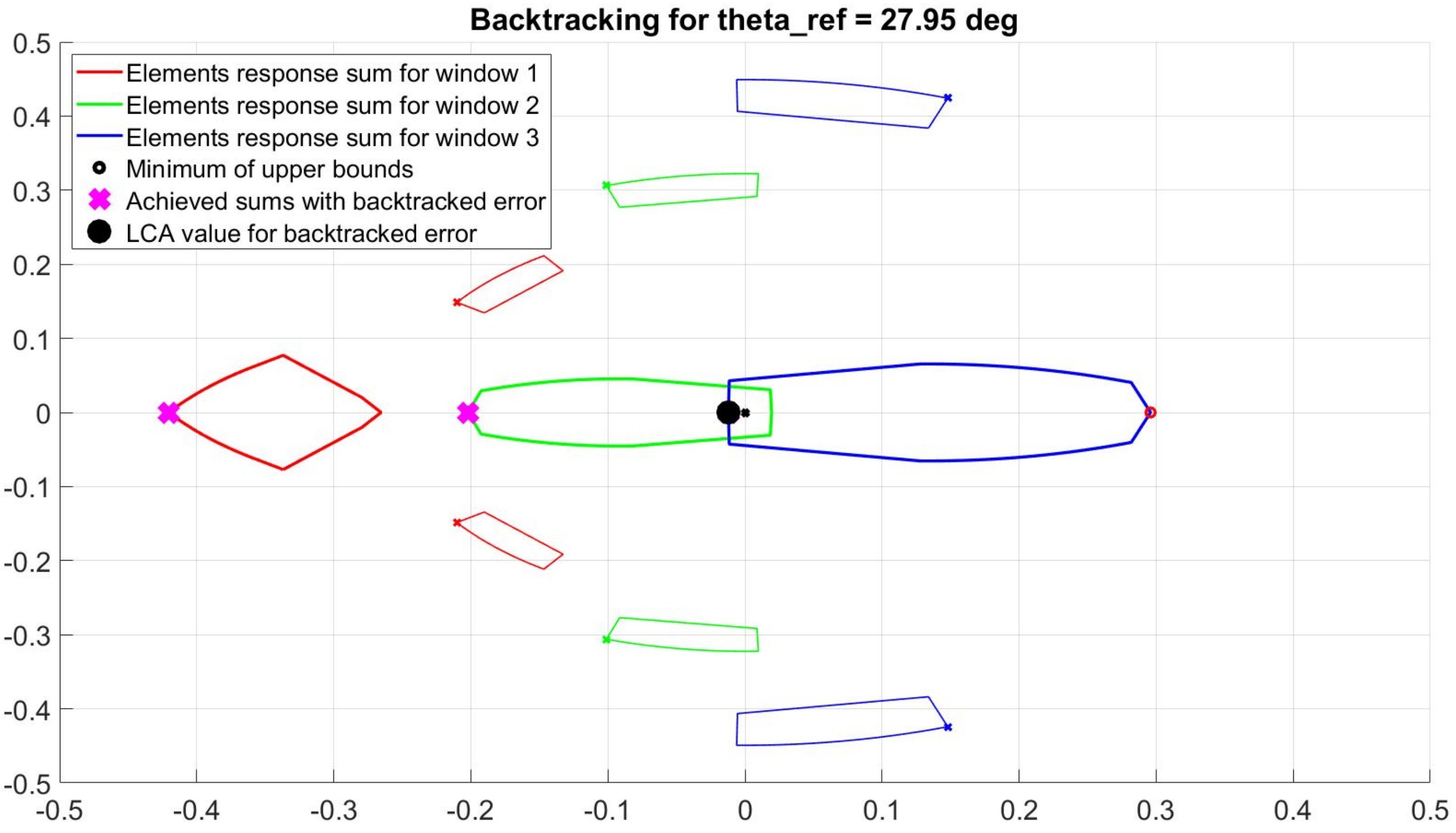
The IA minimum function doesn’t take the dependency into account, hence the relaxed bound

# Trying to get rid of dependency : backtracking

Note that the lower bound is actually tight ! (Due to the commutativity of minimums)

Backtracking aims to recover the elements that sums to an extreme value in an interval (in our case).

Arnestad HK, Geréb G, Lønmo TIB, Kirkebø JE, Austeng A, Näsholm SP. Worst-case analysis of array beampatterns using interval arithmetic. J Acoust Soc Am. 2023 Jun 1;153(6):3312. doi: 10.1121/10.0019715. PMID: 37318184.




Two blocks array - Three windows : maximum values and backtracked errors

# Why backtracking won't work

Backtracked errors are most likely different for every window

"The" error achieving the IA upper bound can result in a completely different result for other windows

The maximum is actually achieved in the interior of the intervals